

# ULTRA $LI$ -IDEALS IN LATTICE IMPLICATION ALGEBRAS AND MTL-ALGEBRAS \*

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**Abstract.** A mistake concerning the ultra  $LI$ -ideal of a lattice implication algebra is pointed out, and some new sufficient and necessary conditions for an  $LI$ -ideal to be an ultra  $LI$ -ideal are given. Moreover, the notion of an  $LI$ -ideal is extended to MTL-algebras, the notions of a (prime, ultra, obstinate, Boolean)  $LI$ -ideal and an  $ILI$ -ideal of an MTL-algebra are introduced, some important examples are given, and the following notions are proved to be equivalent in MTL-algebra: (1) prime proper  $LI$ -ideal and Boolean  $LI$ -ideal, (2) prime proper  $LI$ -ideal and  $ILI$ -ideal, (3) proper obstinate  $LI$ -ideal, (4) ultra  $LI$ -ideal.

**Keywords:** lattice implication algebra, MTL-algebra, (prime, ultra, obstinate, Boolean)  $LI$ -ideal,  $ILI$ -ideal

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## 1 Introduction

In order to research a logical system whose propositional value is given in a lattice, Y. Xu proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems (see [15], [17]). In [7], Y. B. Jun et al. proposed the concept of an  $LI$ -ideal of a lattice implication algebra, discussed the relationship between filters and  $LI$ -ideals, and studied how to generate an  $LI$ -ideal by a set. In [11], K. Y. Qin et al. introduced the notion of ultra  $LI$ -ideals in lattice implication algebras, and gave some sufficient and necessary conditions for an  $LI$ -ideal to be ultra  $LI$ -ideal.

The interest in the foundations of fuzzy logic has been rapidly growing recently and several new algebras playing the role of the structures of truth-values have been introduced. P. Hájek introduced the system of basic logic ( $BL$ ) axioms for the fuzzy propositional logic and defined the class of  $BL$ -algebras (see [4]). G. J. Wang proposed a formal deductive system  $L^*$  for fuzzy propositional calculus, and a kind of new algebraic structures, called  $R_0$ -algebras (see [13], [14]). F. Esteva and L. Godo proposed a new formal deductive system  $MTL$ , called the monoidal  $t$ -norm-based logic, intended to cope with left-continuous  $t$ -norms and their residual. The algebraic semantics for  $MTL$  is based on  $MTL$ -algebras (see [3], [5]). It is easy to verify that a lattice implication algebra is an  $MTL$ -algebra. Varieties of  $MTL$ -algebras are described in [10].

This paper is devoted to a discussion of the ultra  $LI$ -ideals, we correct a mistake in [11] and give some new equivalent conditions for an  $LI$ -ideal to be ultra. We also generalize the notion of an  $LI$ -ideal to  $MTL$ -algebras, introduce the notions of a (prime, ultra, obstinate, Boolean)  $LI$ -ideal and an  $ILI$ -ideal of  $MTL$ -algebra, give some important examples, and prove that the following notions are equivalent in an  $MTL$ -algebra: (1) prime proper  $LI$ -ideal and Boolean  $LI$ -ideal, (2) prime proper  $LI$ -ideal and  $ILI$ -ideal, (3) proper obstinate  $LI$ -ideal, (4) ultra  $LI$ -ideal.

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## 2 Preliminaries

**Definition 2.1.** ([17]) By a *lattice implication algebra*  $L$  we mean a bounded lattice  $(L, \vee, \wedge, 0, 1)$  with an order-reversing involution  $'$  and a binary operation  $\rightarrow$  satisfying the following axioms:

- (I1)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$
- (I2)  $x \rightarrow x = 1,$
- (I3)  $x \rightarrow y = y' \rightarrow x',$
- (I4)  $x \rightarrow y = y \rightarrow x = 1 \implies x = y,$
- (I5)  $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$
- (L1)  $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z),$
- (L2)  $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$  for all  $x, y, z \in L.$

We can define a partial ordering  $\leq$  on a lattice implication algebra  $L$  by

$$x \leq y \text{ if and only if } x \rightarrow y = 1.$$

For any lattice implication algebra  $L$ ,  $(L, \vee, \wedge)$  is a distributive lattice and the De Morgan law holds, that is

- (L3)  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$
- (L4)  $(x \wedge y)' = x' \vee y', \quad (x \vee y)' = x' \wedge y'$  for all  $x, y, z \in L.$

**Theorem 2.2.** ([17]) *In a lattice implication algebra  $L$ , the following relations hold:*

- (1)  $0 \rightarrow x = 1, \quad 1 \rightarrow x = x \text{ and } x \rightarrow 1 = 1,$
- (2)  $x' = x \rightarrow 0,$
- (3)  $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z),$
- (4)  $x \vee y = (x \rightarrow y) \rightarrow y,$
- (5)  $x \leq y \text{ implies } y \rightarrow z \leq x \rightarrow z \text{ and } z \rightarrow x \leq z \rightarrow y,$
- (6)  $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z),$
- (7)  $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z),$
- (8)  $(x \rightarrow y) \vee (y \rightarrow x) = 1,$
- (9)  $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z),$
- (10)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$
- (11)  $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y.$

From the above theorem it follows that lattice implication algebras are strictly connected with *BCC*-algebras and *BCK*-algebras of the form  $(L, \rightarrow, 1)$  [2].

For shortness, in the sequel the formula  $(x \rightarrow y')'$  will be denoted by  $x \otimes y$ , the formula  $x' \rightarrow y$  by  $x \oplus y$ .

**Theorem 2.3.** ([17]) *In a lattice implication algebra  $L$ , the relations*

- (12)  $x \otimes y = y \otimes x, \quad x \oplus y = y \oplus x,$
- (13)  $x \otimes (y \otimes z) = (x \otimes y) \otimes z, \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z,$

- (14)  $x \otimes x' = 0, \quad x \oplus x' = 1,$   
(15)  $x \otimes (x \rightarrow y) = x \wedge y,$   
(16)  $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z,$   
(17)  $x \leq y \rightarrow z \iff x \otimes y \leq z,$   
(18)  $x \leq a \text{ and } y \leq b \text{ imply } x \otimes y \leq a \otimes b \text{ and } x \oplus y \leq a \oplus b$

hold for all  $x, y, z \in L$ .

**Definition 2.4.** ([7]) A subset  $A$  of a lattice implication algebra  $L$  is called an *LI-ideal* of  $L$  if

- (LI1)  $0 \in A,$   
(LI2)  $(x \rightarrow y)' \in A$  and  $y \in A$  imply  $x \in A$  for all  $x, y \in L$ .

An *LI-ideal*  $A$  of a lattice implication algebra  $L$  is said to be *proper* if  $A \neq L$ .

**Theorem 2.5.** ([7], [17]) Let  $A$  be an *LI-ideal* of a lattice implication algebra  $L$ , then

- (LI3)  $x \in A, y \leq x$  imply  $y \in A,$   
(LI4)  $x, y \in A$  imply  $x \vee y \in A.$

The least *LI-ideal* containing a subset  $A$  is called the *LI-ideal generated by  $A$*  and is denoted by  $\langle A \rangle$ .

**Theorem 2.6.** ([7], [17]) If  $A$  is a non-empty subset of a lattice implication algebra  $L$ , then

$$\langle A \rangle = \{x \in L \mid a'_n \rightarrow (\dots \rightarrow (a'_1 \rightarrow x') \dots) = 1 \text{ for some } a_1, \dots, a_n \in A\}.$$

**Theorem 2.7.** ([11]) Let  $A$  be a subset of a lattice implication algebra  $L$ . Then  $A$  is an *LI-ideal* of  $L$  if and only if it satisfies (LI3) and

- (LI5)  $x \in A$  and  $y \in A$  imply  $x \oplus y \in A.$

**Theorem 2.8.** ([11]) If  $A$  is a non-empty subset of a lattice implication algebra  $L$ , then

$$\langle A \rangle = \{x \in L \mid x \leq a_1 \oplus a_2 \oplus \dots \oplus a_n \text{ for some } a_1, \dots, a_n \in A\}.$$

**Definition 2.9.** ([11]) An *LI-ideal*  $A$  of a lattice implication algebra  $L$  is said to be *ultra* if for every  $x \in L$ , the following equivalence holds:

- (LI6)  $x \in A \iff x' \notin A.$

**Definition 2.10.** ([9]) A non-empty subset  $A$  of a lattice implication algebra  $L$  is said to be an *ILI-ideal* of  $L$  if it satisfies (LI1) and

- (LI7)  $((x \rightarrow y)' \rightarrow y)' \rightarrow z' \in A$  and  $z \in A$  imply  $(x \rightarrow y)' \in A$  for all  $x, y, z \in L$ .

**Theorem 2.11.** ([9]) If  $A$  is an *LI-ideal* of a lattice implication algebra  $L$ , then the following assertions are equivalent:

- (i)  $A$  is an *ILI-ideal* of  $L$ ,
- (ii)  $((x \rightarrow y)' \rightarrow y)' \in A$  implies  $(x \rightarrow y)' \in A$  for all  $x, y, z \in L$ ,
- (iii)  $((x \rightarrow y)' \rightarrow z)' \in A$  implies  $((x \rightarrow z)' \rightarrow (y \rightarrow z)')' \in A$  for all  $x, y, z \in L$ ,
- (iv)  $(x \rightarrow (y \rightarrow x)')' \in A$  implies  $x \in A$  for all  $x, y, z \in L$ .

**Definition 2.12.** ([6]) A proper *LI*-ideal  $A$  of a lattice implication algebra  $L$  is said to be a *prime LI*-ideal of  $L$  if  $x \wedge y \in A$  implies  $x \in A$  or  $y \in A$  for any  $x, y \in L$ .

**Theorem 2.13.** ([9]) Let  $A$  be a proper *LI*-ideal of a lattice implication algebra  $L$ . The following assertions are equivalent:

- (i)  $A$  is a prime *LI*-ideals of  $L$ ,
- (ii)  $x \wedge y = 0$  implies  $x \in A$  or  $y \in A$  for any  $x, y \in L$ .

An *LI*-ideal of a lattice implication algebra  $L$  is called *maximal*, if it is proper and not a proper subset of any proper *LI*-ideal of  $L$ .

**Theorem 2.14.** ([9]) In a lattice implication algebra  $L$ , any maximal *LI*-ideal must be prime.

**Theorem 2.15.** ([9]) Let  $L$  be a lattice implication algebra and  $A$  a proper *LI*-ideal of  $L$ . Then  $A$  is both a prime *LI*-ideal and an *ILI*-ideal of  $L$  if and only if  $x \in A$  or  $x' \in A$  for any  $x \in L$ .

**Theorem 2.16.** ([9]) Let  $L$  be a lattice implication algebra and  $A$  a proper *LI*-ideal. Then  $A$  is both a maximal *LI*-ideal and an *ILI*-ideal if and only if for any  $x, y \in L$ ,  $x \notin A$  and  $y \notin A$  imply  $(x \rightarrow y)' \in A$  and  $(y \rightarrow x)' \in A$ .

**Definition 2.17.** ([1], [3]) A *residuated lattice* is an algebra  $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  with four binary operations and two constants such that

- (i)  $(L, \wedge, \vee, 0, 1)$  is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering  $\leq$ ),
- (ii)  $(L, \otimes, 1)$  is a commutative semigroup with the unit element 1, i.e.,  $\otimes$  is commutative, associative,  $1 \otimes x = x$  for all  $x$ ,
- (iii)  $\otimes$  and  $\rightarrow$  form an adjoint pair, i.e.,  $z \leq x \rightarrow y$  if and only if  $z \otimes x \leq y$  for all  $x, y, z \in L$ .

**Definition 2.18.** ([3]) A residuated lattice  $L$  is called an *MTL*-algebra, if it satisfies the pre-linearity equation:  $(x \rightarrow y) \vee (y \rightarrow x) = 1$  for all  $x, y \in L$ . An *MTL*-algebra  $L$  is called an *IMTL*-algebra, if  $(a \rightarrow 0) \rightarrow 0 = a$  for any  $a \in L$ .

In the sequel  $x'$  will be reserved for  $x \rightarrow 0$ ,  $L$  for  $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ .

**Proposition 2.19.** ([3], [12]) Let  $L$  be a residuated lattice. Then for all  $x, y, z \in L$ ,

- (R1)  $x \leq y \iff x \rightarrow y = 1$ ,
- (R2)  $x = 1 \rightarrow x$ ,  $x \rightarrow (y \rightarrow x) = 1$ ,  $y \leq (y \rightarrow x) \rightarrow x$ ,
- (R3)  $x \leq y \rightarrow z \iff y \leq x \rightarrow z$ ,
- (R4)  $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z)$ ,
- (R5)  $x \leq y$  implies  $z \rightarrow x \leq z \rightarrow y$  and  $y \rightarrow z \leq x \rightarrow z$ ,
- (R6)  $z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y)$ ,  $z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x)$ ,
- (R7)  $(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$ ,
- (R8)  $x' = x'''$ ,  $x \leq x''$ ,
- (R9)  $x' \wedge y' = (x \vee y)'$ ,
- (R10)  $x \vee x' = 1$  implies  $x \wedge x' = 0$ ,
- (R11)  $(\bigvee_{i \in \Gamma} y_i) \rightarrow x = \bigwedge_{i \in \Gamma} (y_i \rightarrow x)$ ,

$$(R12) \quad x \otimes (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma} (x \otimes y_i),$$

$$(R13) \quad x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i),$$

$$(R14) \quad \bigvee_{i \in \Gamma} (y_i \rightarrow x) \leq (\bigwedge_{i \in \Gamma} y_i) \rightarrow x,$$

where  $\Gamma$  is a finite or infinite index set and we assume that the corresponding infinite meets and joints exist in  $L$ .

**Proposition 2.20.** ([3], [18]) Let  $L$  be an *MTL-algebra*. Then for all  $x, y, z \in L$ ,

$$(M1) \quad x \otimes y \leq x \wedge y,$$

$$(M2) \quad x \leq y \text{ implies } x \otimes z \leq y \otimes z,$$

$$(M3) \quad y \rightarrow z \leq x \vee y \rightarrow x \vee z,$$

$$(M4) \quad x' \vee y' = (x \wedge y)',$$

$$(M5) \quad (x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z),$$

$$(M6) \quad x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x),$$

$$(M7) \quad x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z),$$

$$(M8) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \text{ i.e., the lattice structure of } L \text{ is distributive.}$$

**Definition 2.21.** ([3]) Let  $L$  be an *MTL-algebra*. A *filter* is a nonempty subset  $F$  of  $L$  such that

$$(F1) \quad x \otimes y \in F \text{ for any } x, y \in F,$$

$$(F2) \quad \text{for any } x \in F \text{ } x \leq y \text{ implies } y \in F.$$

**Proposition 2.22.** ([3]) A subset  $F$  of an *MTL-algebra*  $L$  is a filter of  $L$  if and only if

$$(F3) \quad 1 \in F,$$

$$(F4) \quad x \in F \text{ and } x \rightarrow y \in F \text{ imply } y \in F.$$

### 3 Ultra *LI*-ideals of lattice implication algebras

In [11], the following result is presented: let  $A$  be a subset of a lattice implication algebra  $L$ , then  $A$  is an ultra *LI-ideal* of  $L$  if and only if  $A$  is a maximal proper *LI-ideal* of  $L$ . The following example shows that this result is not true.

**Example 3.1.** Let  $L = \{0, a, b, 1\}$  be a set with the Cayley table

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any  $x \in L$ , we have  $x' = x \rightarrow 0$ . The operations  $\wedge$  and  $\vee$  on  $L$  are defined as follows:

$$x \vee y = (x \rightarrow y) \rightarrow y, \quad x \wedge y = ((x' \rightarrow y') \rightarrow y')'.$$

Then  $(L, \vee, \wedge, 0, 1)$  is a lattice implication algebra. It is easy to check that  $\{0\}$  is a maximal proper *LI-ideal* of  $L$ , but not an ultra *LI-ideal* of  $L$ , because  $a' = b \notin \{0\}$ , but  $a \notin \{0\}$ .

Below, we give some new sufficient and necessary conditions for an  $LI$ -ideal to be an ultra  $LI$ -ideal.

**Theorem 3.2.** *Let  $L$  be a lattice implication algebra and  $A$  an  $LI$ -ideal of  $L$ . Then the following assertions are equivalent:*

- (i)  $A$  is an ultra  $LI$ -ideal,
- (ii)  $A$  is a prime proper  $LI$ -ideal and an  $ILI$ -ideal,
- (iii)  $A$  is a proper  $LI$ -ideal and  $x \in A$  or  $x' \in A$  for any  $x \in L$ ,
- (iv)  $A$  is a maximal  $ILI$ -ideal,
- (v)  $A$  is a proper  $LI$ -ideal and for any  $x, y \in L$ ,  $x \notin A$  and  $y \notin A$  imply  $(x \rightarrow y)' \in A$  and  $(y \rightarrow x)' \in A$ .

*Proof.* (i)  $\implies$  (ii):  $A$  is a proper  $LI$ -ideal, because  $0 \in A$ , and so  $1 = 0' \notin A$ .

We show that  $A$  is a prime  $LI$ -ideal. Assume  $x \wedge y = 0$  for some  $x, y \in L$ . We prove that  $x \in A$  or  $y \in A$ . If  $x \notin A$  and  $y \notin A$ , then  $x' \in A$  and  $y' \in A$ , by the definition of an ultra  $LI$ -ideal. So, by Theorem 2.5 (LI4), we have  $x' \vee y' \in A$ , thus  $1 = 0' = (x \wedge y)' = x' \vee y' \in A$ . This means that  $A = L$ , a contradiction. Therefore  $x \wedge y = 0$  implies  $x \in A$  or  $y \in A$ . So, by Theorem 2.13,  $A$  is a prime proper  $LI$ -ideal.

Now we show that  $A$  is an  $ILI$ -ideal. Let  $((x \rightarrow y)' \rightarrow y)' \in A$ . If  $(x \rightarrow y)' \notin A$ , then  $x \rightarrow y \in A$  by the definition of an ultra  $LI$ -ideal. Since  $y \leq x \rightarrow y$ , we have  $y \in A$ . From  $((x \rightarrow y)' \rightarrow y)' \in A$  and  $y \in A$ , we conclude  $(x \rightarrow y)' \in A$ , by Definition 2.4 (LI2). This is a contradiction. Thus,  $(x \rightarrow y)' \in A$ . By Theorem 2.11 (ii),  $A$  is an  $ILI$ -ideal. This means that (ii) holds.

(ii)  $\iff$  (iii): See Theorem 2.15.

(iii)  $\implies$  (i): For any  $x \in L$ , if  $x' \notin A$  then  $x \in A$  by (iii). If  $x \in A$ , we prove that  $x' \notin A$ . Indeed, if  $x' \in A$ , then  $x \oplus x' = 1 \in A$  by Theorem 2.3 (14) and Theorem 2.7 (LI5). This is a contradiction with the fact that  $A$  is a proper  $LI$ -ideal. This means that  $A$  is an ultra  $LI$ -ideal.

(iv)  $\iff$  (v): See Theorem 2.16.

(i)  $\implies$  (v):  $A$  is a proper  $LI$ -ideal, because  $0 \in A$ , and so  $1 = 0' \notin A$ .

If  $x \notin A$ , from  $x \leq y \rightarrow x$  and Theorem 2.7 (LI3), we have  $y \rightarrow x \notin A$ . Thus, by the definition of an ultra  $LI$ -ideal,  $(y \rightarrow x)' \in A$ . Similarly, from  $y \notin A$  we obtain  $(x \rightarrow y)' \in A$ . That is, (v) holds.

(v)  $\implies$  (i): By (v),  $1 \notin A$ . If  $x' \notin A$ , by (v) we have  $(1 \rightarrow x')' \in A$ , that is  $x \in A$ . If  $x \in A$ , then  $x' \notin A$  (see (iii)  $\implies$  (i)). This means that  $A$  is an ultra  $LI$ -ideal. The proof is complete.  $\square$

Remind [11] that a subset  $A$  of a lattice implication algebra  $L$  has the *finite additive property* if  $a_1 \oplus a_2 \oplus \dots \oplus a_n \neq 1$  for any finite members  $a_1, \dots, a_n \in A$ .  $\langle A \rangle$  is a proper  $LI$ -ideal of  $L$  if and only if  $A$  has the finite additive property.

Our theorem proves that the part results formulated in Theorem 3.7 and Corollary 3.8 in [11] is correct. Namely we have

**Theorem 3.3.** *If a subset  $A$  of a lattice implication algebra  $L$  has the finite additive property, then there exists a maximal  $LI$ -ideal of  $L$  containing  $A$ . Every proper  $LI$ -ideal of a lattice implication algebra can be extended to a maximal  $LI$ -ideal.*

## 4 $LI$ -ideals of MTL-algebras

**Definition 4.1.** A subset  $A$  of an  $MTL$ -algebra  $L$  is called an  $LI$ -ideal of  $L$  if  $0 \in A$  and

(LI8)  $(x' \rightarrow y')' \in A$  and  $x \in A$  imply  $y \in A$  for all  $x, y \in L$ .

Obviously, for a lattice implication algebra  $L$ , (LI8) coincides with (LI2). For a  $MTL$ -algebra it is not true because  $x = x''$  is not true.

An  $LI$ -ideal  $A$  of an  $MTL$ -algebra  $L$  is said to be *proper* if  $A \neq L$ .

**Lemma 4.2.** ([17] Theorem 4.1.3) *A non-empty subset  $A$  of a lattice implication algebra  $L$  is a filter of  $L$  if and only if  $A' = \{a' \mid a \in A\}$  is an  $LI$ -ideal of  $L$ .*

For  $MTL$ -algebras the above lemma is not true.

**Example 4.3.** Consider the set  $L = \{0, a, b, c, d, 1\}$ , where  $0 < a < b < c < d < 1$ , and two operations  $\otimes, \rightarrow$  defined by the following two tables:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	b	b	b
c	0	0	b	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

If we define on  $L$  the operations  $\wedge$  and  $\vee$  as min and max, respectively, then  $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  will be an  $MTL$ -algebra. Obviously,  $A = \{0, a, b, c, d, 1\}$  is a filter of  $L$ , but  $A' = \{0, a, b, c, 1\}$  is not an  $LI$ -ideal of  $L$ , since

$$(0' \rightarrow d')' = 1 \in A \text{ and } 0 \in A, \text{ but } d \notin A.$$

Moreover,  $B = \{1, c\}$  is not a filter of  $L$ , because  $c \rightarrow d = 1 \in B$  and  $c \in B$ ,  $d \notin B$ . By the following MATHEMATICA program, we can verify that  $B' = \{0, a\}$  is an  $LI$ -ideal of  $L$ :

```

M1={ {6,6,6,6,6,6},{4,6,6,6,6,6},{3,3,6,6,6,6},{2,2,3,6,6,6},{1,2,3,4,6,6},{1,2,3,4,5,6}};
a1=0;
For[i=1, i<7, i++, For[j=1, j<7, j++,
  If[(i==1||i==2) && (M1[[M1[[M1[[i,1]],M1[[j,1]]],1]]==1||
    M1[[M1[[M1[[i,1]],M1[[j,1]]],1]]==2) && (j!=1&&j!=2), a1++]];
If[a1==0, Print["true"], Print["false"]]

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From Example 4.3 we see that  $LI$ -ideals have a proper meaning in  $MTL$ -algebras.

**Theorem 4.4.** *Let  $A$  be an  $LI$ -ideal of an  $MTL$ -algebra  $L$ , then*

(LI3) *if  $x \in A$ ,  $y \leq x$ , then  $y \in A$ ,*

(LI9) *if  $x \in A$ , then  $x'' \in A$ ,*

(LI4) *if  $x, y \in A$ , then  $x \vee y \in A$ .*

*Proof.* Assume  $x \in A$ ,  $y \leq x$ . From  $y \leq x$ , by Proposition 2.19 (R5), we have  $x \rightarrow 0 \leq y \rightarrow 0$ , i.e.,  $x' \leq y'$ . By Proposition 2.19 (R1),  $x' \rightarrow y' = 1$ . Then  $(x' \rightarrow y')' = 1' = 0 \in A$  and  $x \in A$ , and by (LI8) we get  $y \in A$ . This means that (LI3) holds.

Suppose  $x \in A$ . By Proposition 2.19 (R8) we have  $(x' \rightarrow (x''))' = (x' \rightarrow x')' = 1' = 0 \in A$ . Applying (LI8) we get  $x'' \in A$ , i.e., (LI9) holds.

Assume  $x, y \in A$ . By Proposition 2.19 (R2) we have  $y' \leq x' \rightarrow y'$ . So,  $(x' \rightarrow y')' \leq y''$  by (R5). Whence, by  $y \in A$  and (LI9), we obtain  $y'' \in A$ . From this and  $(x' \rightarrow y')' \leq y''$ , using (LI3) we get  $(x' \rightarrow y')' \in A$ . Thus

$$\begin{aligned}
(x' \rightarrow (x \vee y)')' &= (x' \rightarrow (x' \wedge y'))' && \text{(by (R9))} \\
&= ((x' \rightarrow x') \wedge (x' \rightarrow y'))' && \text{(by (R13))} \\
&= (1 \wedge (x' \rightarrow y'))' && \text{(by (R1))} \\
&= (x' \rightarrow y')' \in A.
\end{aligned}$$

From this and  $x \in A$ , using (LI8), we deduce  $x \vee y \in A$ , i.e., (LI4) holds.

The proof is complete.  $\square$

**Definition 4.5.** An  $LI$ -ideal  $A$  of an  $MTL$ -algebra  $L$  is said to be an  $ILI$ -ideal of  $L$  if it satisfies

(LI10)  $(x \rightarrow (y \rightarrow x)')' \in A$  implies  $x \in A$  for all  $x, y, z \in L$ .

**Example 4.6.** Let  $L = \{0, a, b, 1\}$ , where  $0 < a < b < 1$ , be a set with the Cayley tables:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Defining the operations  $\wedge$  and  $\vee$  on  $L$  as min and max, respectively, we obtain an  $MTL$ -algebra  $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  in which  $A = \{0\}$  is an  $ILI$ -ideal of  $L$ .

In Example 4.3,  $\{0, a\}$  is an  $LI$ -ideal, but it is not an  $ILI$ -ideal of  $L$ , because

$$(b \rightarrow (1 \rightarrow b)')' = 0 \in \{0, a\}, \text{ but } b \notin \{0, a\}.$$

**Theorem 4.7.** For each an  $ILI$ -ideal  $A$  of an  $MTL$ -algebra  $L$  we have

(LI11)  $x \wedge x' \in A$  for each  $x \in L$ .

*Proof.* Indeed, for all  $x \in L$  we get

$$\begin{aligned}
((x \wedge x') \rightarrow (1 \rightarrow (x \wedge x')')')' &= ((x \wedge x') \rightarrow (x \wedge x')')' \\
&= ((x \wedge x') \rightarrow (x' \vee x''))' && \text{(by Proposition 2.20 (M4))} \\
&= (((x \wedge x') \rightarrow x') \vee ((x \wedge x') \rightarrow x''))' && \text{(by Proposition 2.20 (M7))} \\
&= (1 \vee ((x \wedge x') \rightarrow x''))' && \text{(by Proposition 2.19 (R1))} \\
&= 1' = 0 \in A.
\end{aligned}$$

From this, applying (LI10), we deduce (LI11).  $\square$

**Definition 4.8.** An  $LI$ -ideal  $A$  satisfying (LI11) is called *Boolean*.

**Theorem 4.9.** If  $A$  is a Boolean  $LI$ -ideal of an  $MTL$ -algebra  $L$ , then

(LI12)  $(x \rightarrow x')' \in A$  implies  $x \in A$ .

*Proof.* According to the assumption  $x \wedge x' \in A$  for all  $x \in L$ . Let  $(x \rightarrow x')' \in A$ . Then

$$\begin{aligned}
((x \wedge x')' \rightarrow x')' &= (x \rightarrow (x \wedge x')'')' && \text{(by Proposition 2.19 (R4))} \\
&= (x \rightarrow (x'' \wedge x'''))' && \text{(by Propositions 2.19 (R9) and 2.20 (M4))} \\
&= ((x \rightarrow x'') \wedge (x \rightarrow x'''))' && \text{(by Proposition 2.19 (R13))} \\
&= (1 \wedge (x \rightarrow x'))' && \text{(by Proposition 2.19 (R8), (R1))} \\
&= (x \rightarrow x')' \in A.
\end{aligned}$$

Now, applying (LI8) we get  $x \in A$ , which completes the proof.  $\square$

**Theorem 4.10.** For  $LI$ -ideals of  $MTL$ -algebras the conditions (LI10) are equivalent (LI11).



*Proof.* (LI10) $\implies$ (LI11): See Theorem 4.7.

(LI11) $\implies$ (LI10): Let  $(x \rightarrow (y \rightarrow x)')' \in A$ . Then

$$\begin{aligned}
((x \rightarrow (y \rightarrow x)')'' \rightarrow (x \rightarrow x')'')' &= ((x \rightarrow x')' \rightarrow (x \rightarrow (y \rightarrow x)')')' && \text{(by Proposition 2.19 (R4), (R8))} \\
&\leq ((x \rightarrow (y \rightarrow x)') \rightarrow (x \rightarrow x'))' && \text{(by Proposition 2.19 (R6))} \\
&\leq ((y \rightarrow x)' \rightarrow x')' && \text{(by Proposition 2.19 (R6))} \\
&\leq (x \rightarrow (y \rightarrow x))' && \text{(by Proposition 2.19 (R6))} \\
&= 1' = 0 \in A. && \text{(by Proposition 2.19 (R2))}
\end{aligned}$$

This, by (LI8), implies  $(x \rightarrow x')' \in A$ , whence, using (LI12), we obtain  $x \in A$ . So, (LI11) implies (LI10).  $\square$

**Definition 4.11.** A proper *LI*-ideal  $A$  of an *MTL*-algebra  $L$  is said to be a *it prime* if  $x \wedge y \in A$  implies  $x \in A$  or  $y \in A$  for any  $x, y \in L$ .

**Theorem 4.12.** A proper *LI*-ideal  $A$  of a *MTL*-algebra  $L$  is prime if and only if for all  $x, y \in L$  we have  $(x \rightarrow y)' \in A$  or  $(y \rightarrow x)' \in A$ .

*Proof.* Assume that an *LI*-ideal  $A$  of  $L$  is prime. Since

$$(x \rightarrow y)' \wedge (y \rightarrow x)' = ((x \rightarrow y) \vee (y \rightarrow x))' = 1' = 0 \in A$$

for all  $x, y \in L$ , the assumption on  $A$  implies  $(x \rightarrow y)' \in A$  or  $(y \rightarrow x)' \in A$ .

Conversely, let  $A$  be a proper *LI*-ideal of  $L$  and let  $x \wedge y \in A$ . Assume that  $(x \rightarrow y)' \in A$  or  $(y \rightarrow x)' \in A$  for  $x, y \in L$ . If  $(x \rightarrow y)' \in A$ , then

$$\begin{aligned}
((x \wedge y)' \rightarrow x')' &= ((x' \vee y') \rightarrow x')' && \text{(by Proposition 2.20 (M4))} \\
&= ((x' \rightarrow x') \wedge (y' \rightarrow x'))' && \text{(by Proposition 2.19 (R11))} \\
&= (1 \wedge (y' \rightarrow x'))' && \text{(by Proposition 2.19 (R1))} \\
&= (y' \rightarrow x')' \leq (x \rightarrow y)' \in A. && \text{(by Proposition 2.19 (R6))}
\end{aligned}$$

So,  $((x \wedge y)' \rightarrow x')' \in A$  (Theorem 4.4 (LI3)), which together with  $x \wedge y \in A$  and the definition of an *LI*-ideal, gives  $x \in A$ .

Similarly, from  $(y \rightarrow x)' \in A$  we can obtain  $y \in A$ .

This means that  $A$  is a prime *LI*-ideal of  $L$ . The proof is complete.  $\square$

**Theorem 4.13.** Let  $A$  be an *LI*-ideal of an *MTL*-algebra  $L$ . Then  $A$  is both a prime *LI*-ideal and a Boolean *LI*-ideal of  $L$  if and only if  $x \in A$  or  $x' \in A$  for any  $x \in L$ .

*Proof.* Assume that for all  $x \in L$  we have  $x \in A$  or  $x' \in A$ . At first we show that an *LI*-ideal  $A$  is prime. For this let  $x \wedge y \in A$ . If  $x \notin A$ , then  $x' \in A$ . Hence

$$\begin{aligned}
((x \wedge y)' \rightarrow y')' &= ((x' \vee y') \rightarrow y')' && \text{(by Proposition 2.20 (M4))} \\
&= ((x' \rightarrow y') \wedge (y' \rightarrow y'))' && \text{(by Proposition 2.19 (R11))} \\
&= ((x' \rightarrow y') \wedge 1)' && \text{(by Proposition 2.19 (R1))} \\
&= (x' \rightarrow y')' \leq (y \rightarrow x)' && \text{(by Proposition 2.19 (R6))} \\
&\leq x' \in A. && \text{(by Proposition 2.19 (R2))}
\end{aligned}$$

So,  $((x \wedge y)' \rightarrow y')' \in A$ , by Theorem 4.4 (LI3). From this,  $x \wedge y \in A$  and Definition 4.1 we get  $y \in A$ . This proves that an *LI*-ideal  $A$  is prime. To prove that it is Boolean observe that  $x \wedge x' \leq x'$  implies  $x \wedge x' \leq x$ , whence, by Theorem 4.4 (LI3), we obtain  $x \wedge x' \in A$ . Thus  $A$  is Boolean.

Conversely, if an *LI*-ideal  $A$  is both prime and Boolean, then by Definition 4.8, for all  $x \in L$  we have  $x \wedge x' \in A$ . Hence  $x \in A$  or  $x' \in A$ , by Definition 4.11. This completes the proof.  $\square$

**Definition 4.14.** An  $LI$ -ideal  $A$  of an  $MTL$ -algebra  $L$  is said to be *ultra* if for every  $x \in L$

$$(LI6) \quad x \in A \iff x' \notin A.$$

It is easy to verify the following proposition is true.

**Proposition 4.15.** *Each ultra  $LI$ -ideal of an  $MTL$ -algebra is a proper  $LI$ -ideal.*

**Definition 4.16.** An  $LI$ -ideal  $A$  of an  $MTL$ -algebra  $L$  is said to be *obstinate* if for all  $x, y \in L$

$$(LI13) \quad x \notin A \text{ and } y \notin A \text{ imply } (x \rightarrow y)' \in A \text{ and } (y \rightarrow x)' \in A.$$

**Theorem 4.17.** *For an  $LI$ -ideal  $A$  of an  $MTL$ -algebra  $L$  the following conditions are equivalent:*

- (i)  $A$  is an ultra  $LI$ -ideal,
- (ii)  $A$  is a proper  $LI$ -ideal and for any  $x \in L$ ,  $x \in A$  or  $x' \in A$ ,
- (iii)  $A$  is a prime proper  $LI$ -ideal and a Boolean  $LI$ -ideal,
- (iv)  $A$  is a prime proper  $LI$ -ideal and an  $ILI$ -ideal,
- (v)  $A$  is a proper obstinate  $LI$ -ideal.

*Proof.* (i)  $\longrightarrow$  (ii): Obvious.

(ii)  $\longrightarrow$  (i): If  $x' \notin A$ , then  $x \in A$ , by (ii). Similarly, if  $x \in A$ , that must be  $x' \notin A$ . If not, i.e., if  $x' \in A$ , then, by Proposition 2.19 (R8), we have

$$(x' \rightarrow 1')' = (x' \rightarrow 0)' = x''' = x' \in A,$$

which together with  $x \in A$  and (LI8) implies  $1 \in A$ . This, by Theorem 4.4 (LI3), gives  $A = L$ . This is a contradiction, because an  $LI$ -ideal  $A$  is proper. Obtained contradiction proves that  $x \in A$  implies  $x' \notin A$ . So,  $A$  is an ultra  $LI$ -ideal.

(ii)  $\iff$  (iii): See Theorem 4.13.

(iv)  $\implies$  (iii): See Theorem 4.7.

(iii)  $\implies$  (iv): See Theorem 4.10.

(v)  $\implies$  (ii): Since  $A$  is a proper  $LI$ -ideal,  $1 \notin A$ . If  $x \notin A$ , then  $(1 \rightarrow x)' = x' \in A$ , by Definition 4.16. This means that (ii) holds.

(ii)  $\implies$  (v): It suffices to show that  $A$  is obstinate. Let  $x \notin A$  and  $y \notin A$ . Then, according to (ii), we have  $x' \in A$  and  $y' \in A$ . Thus

$$\begin{aligned}
(y'' \rightarrow (x \rightarrow y)'')' &= ((x \rightarrow y)' \rightarrow y''')' && \text{(by Proposition 2.19 (R4))} \\
&= ((x \rightarrow y)' \rightarrow y')' && \text{(by Proposition 2.19 (R8))} \\
&= (y \rightarrow (x \rightarrow y)'')' && \text{(by Proposition 2.19 (R4))} \\
&\leq (y \rightarrow (x \rightarrow y))' && \text{(by (R8), } x \rightarrow y \leq (x \rightarrow y)'' \text{ and (R5))} \\
&= 1' = 0 \in A. && \text{(by Proposition 2.19 (R2))}
\end{aligned}$$

This together with  $y' \in A$  and Definition 4.1 implies  $(x \rightarrow y)' \in A$ .

Similarly, we obtain  $(y \rightarrow x)' \in A$ . So,  $A$  is obstinate.

The proof is complete.  $\square$

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